

# Darboux-integrable nonlinear Liouville-von Neumann equation

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A new form of a binary Darboux transformation is used to generate analytical solutions of a nonlinear Liouville-von Neumann equation. General theory is illustrated by explicit examples.

## I. INTRODUCTION

Nonlinear operator equations one encounters in quantum optics and quantum field theory are typically solved by techniques which are either perturbative or semiclassical (cf. [1,2]). The situation is caused by the fact that analytic methods of dealing with “non-Abelian” nonlinearities are still at a rather preliminary stage of development. An important step towards more efficient analytical techniques is associated with the notion of an inverse spectral transformation. The use of the method in the context of matrix equations can be found in [3–5] where an analytical treatment of Maxwell-Bloch equations is given. In application to the Maxwell-Bloch system describing three-level atoms interacting with light [6] one makes use of a degenerate Zakharov-Shabat spectral problem with reduction constraints [7]. The same problem is used in the context of the complex modified Korteweg-de Vries (MKdV) equation for a slowly varying envelope of electromagnetic field in an optical fiber [8].

A technical complication occurs if a solution obtained by an inverse method should additionally satisfy some constraint. For example, it is often essential to guarantee that the solution one gets is Hermitian or positive. Difficulties of this kind were one of the motivations for the development of new Darboux-type operator techniques of solving non-Abelian equations. Particularly useful turned out to be the method of elementary and binary Darboux transformations introduced by one of us [9–11]. These particular versions of the Darboux transformations are more primitive than the ordinary ones [12] in the sense that the latter can be obtained by their composition. The binary transformation, a result of an application of two mutually conjugated elementary Darboux transformations one after another, was successfully applied to a three-state Maxwell-Bloch system with degeneracy in [9], and various multisoliton solutions, including the well known  $2\pi$ -pulse and breathers, were found.

In this article we apply a generalization of this technique [11] to a new type of nonlinear operator equation. The nonlinear Liouville-von Neumann equation we will discuss is the simplest nontrivial example of a Lie-Nambu dynamics of a density matrix and occurs naturally in certain version of nonlinear quantum mechanics. To begin with, let us recall that the well known Liouville-von Neumann equation (LvNE)

$$i\dot{\rho} = [H, \rho], \quad (1)$$

where  $H$  is a Hamiltonian operator,  $\rho$  a density matrix, and the dot denotes the time derivative, is linear. In Hartree-type theories one considers more general, nonlinear equations of the form

$$i\dot{\rho} = [H(\rho), \rho], \quad (2)$$

where  $H(\rho)$  is a nonlinear Hamiltonian operator. For time-independent Hamiltonians  $H(\rho(t)) = H(\rho(0))$  the formal solutions are exponential

$$\rho(t) = \exp[-iH(\rho(0))t]\rho(0)\exp[iH(\rho(0))t]. \quad (3)$$

Both kinds of nonlinear LvNE’s can be written in either Lie-Poisson [13,14] or Lie-Nambu forms [15–19]. The Lie-Nambu version involves a 3-bracket and the LvNE’s can be written as

$$i\dot{\rho}_a = \{\rho_a, H_f, S\} = \{\rho_a, H_f\} \quad (4)$$

where  $\{\cdot, \cdot\} := \{\cdot, \cdot, S\}$  is a Lie-Poisson bracket. Here  $\rho_a := \rho_{AA'}(\mathbf{a}, \mathbf{a}')$  are components of  $\rho$  in some basis,  $A$  and  $A'$  are discrete (say, spinor) indices and  $\mathbf{a}, \mathbf{a}'$  the continuous ones.  $H_f$  is a Hamiltonian function and  $S$  a functional that can be identified with the 2-entropy of Daróczy [21] and Tsallis [22], i.e.  $2S = \text{Tr}(\rho^2) = \|\rho\|^2$  is the Hilbert-Schmidt squared norm of  $\rho$ .

An extension from a Lie-Poisson bracket to a 3-bracket led Nambu to a generalization of classical Hamiltonian dynamics [23]. The 3-bracket equation (4) naturally leads to the question of possible Nambu-type extensions of the

Lie-Poisson dynamics of density matrices. An interesting class of such generalizations occurs if one keeps  $H_f(\rho)$  linear in  $\rho$  but  $S$  is a function of other Daróczy-Tsallis entropies. Such Nambu-type equations are rather unusual from the point of view of generalized Nambu-Poisson theories [24–34]. The peculiarity is that although the 3-bracket itself does not satisfy the so-called fundamental identity, typically regarded as a Nambu analogue of the Jacobi identity, the 2-bracket defined via  $\{\cdot, \cdot\}_{H_f} := \{\cdot, H_f, \cdot\}$  does satisfy the ordinary Jacobi identity if  $H_f(\rho)$  is a linear functional of  $\rho$  [18]. It follows that restricting  $H_f$  to linear functionals one effectively uses the Nambu-type structure as an intermediate step which allows one to introduce a new Poisson structure, and now  $S$  plays a role of a Hamiltonian function. The standard way of introducing interactions, i.e. by modifying  $H_f$ , looks from this perspective as a modification of the Poisson structure while keeping the Hamiltonian  $S$  fixed. A simultaneous change of the two Nambu-type generators,  $H_f$  and  $S$ , can be regarded as a change of the Hamiltonian function  $S$ , accompanied by a modification of the Poisson structure defined in terms of  $H_f$ . In this respect the Nambu-type bi-Hamiltonian dynamics with linear  $H_f$  has a logical structure analogous to this of general relativity. The choice of linear  $H_f$  and generalized  $S$  can be also motivated by difficulties with probability interpretation of generalized observables since there is no *physically* natural definition of spectrum of nonlinear operators [16,35,36]. The 3-bracket structure can be shown to be a particular case of a still more general  $(2k+1)$ -bracket one that, for  $2k+1 > 3$ , always vanishes on pure states and therefore is invisible at the level of the Schrödinger dynamics [18].

The nonlinear LvNE corresponding to  $S = \text{Tr}(\rho^n)/n$ ,

$$i\dot{\rho} = [H, \rho^{n-1}], \quad (5)$$

was introduced in [16]. General properties of such equations were discussed in [17] and [18]. It was shown, in particular, that spectra of their Hermitian Hilbert-Schmidt solutions are time-independent. This opens a possibility of a density matrix interpretation of the solutions. Let us note that for  $\rho^2 = \rho$  (pure states) the equations reduce to the linear LvNE and, therefore, the pure state dynamics is indistinguishable from the ordinary linear Schrödinger one. One of the problems that still remained open was how to solve such nonlinear equations. There exist formal solutions given in a form of a series, but the question of convergence of such a series was not investigated.

The aim of this paper is to describe an algebraic method that leads to solutions of a nonlinear LvNE which reduces to (5) with  $n = 3$  (3-entropy equation) in special cases. The equation we shall study is

$$i\dot{\rho} = [H, \rho^2] + i\rho'H + iH\rho' \quad (6)$$

where the prime denotes a derivative with respect to some additional parameter  $\tau$ . We will generate the solutions from a Lax pair with the help of a binary Darboux transformation. To avoid technicalities we will generally assume that the Hamiltonian  $H$  and other operators are finite-dimensional matrices, but the transformation works in a much more general setting (see the example of the harmonic oscillator) and its application to general infinite-dimensional systems is a subject of current study.

## II. LAX PAIR AND ITS DARBOUX COVARIANCE

The technique of Darboux-type transformations is perhaps the most powerful analytical method of solving differential equations. Although it was developed mainly in the context of nonlinear equations, it is implicitly used also in standard textbook quantum mechanics under the name of the creation-annihilation operator method. The method of creation operators is simultaneously a good illustration of the way the Darboux technique works. In short, to use the method one has to begin with an initial solution which is found by other means (a “ground state”). Then one has to find a “creation operator” and the Darboux transformation is a systematic procedure that allows one to do it. In linear cases once we have these two elements, we are able to generate an entire Hilbert space of solutions. In nonlinear cases the spaces of solutions are bigger and therefore a given “ground state” and a “creation operator” may generate only a subset of this space. It is mainly for this reason that much effort was devoted to finding different generalizations of Darboux transformations (cf. [12]). The method we will use was devised for non-commutative equations such as Heisenberg equations of motion. The construction given in [10,11] led to a transformation more general than the one we use and its derivation from elementary transformations is somewhat tedious. However, once one has our explicit form, one can check by a straightforward calculation that the binary transformation indeed maps one solution into another. To make this paper self-contained we give the explicit proof in the Appendix.

Consider the following pair of Zakharov-Shabat equations

$$i\varphi(\mu)' = (U - \mu H)\varphi(\mu) =: Z_\mu\varphi(\mu) \quad (7)$$

$$i\dot{\varphi}(\mu) = (UH + HU - \mu H^2)\varphi(\mu) \quad (8)$$

$$= \frac{1}{\mu}(U^2 - Z_\mu^2)\varphi(\mu) \quad (9)$$

where  $U$  and  $H$  are Hermitian matrices the dot and prime denote, respectively, derivatives with respect to time  $t$  and some auxiliary parameter  $\tau$ , and  $\mu$  is complex. The solution  $\varphi(\mu)$  is also in general a matrix. We assume that  $H$  is  $t$  and  $\tau$ -independent and  $U = U(t, \tau)$ . The compatibility condition for (7), (8) is

$$i\dot{U} = [H, U^2] + iU'H + iHU', \quad (10)$$

and therefore the above pair is the Lax pair for (6). We will stick to the notation with  $U$  instead of  $\rho$  since non-Hermitian and non-positive solutions are also of some interest and  $\rho$  will be reserved for density matrices.

We will need two additional conjugated problems

$$-i\psi(\lambda)' = \psi(\lambda)(U - \lambda H) \quad (11)$$

$$-i\dot{\psi}(\lambda) = \psi(\lambda)(UH + HU - \lambda H^2) \quad (12)$$

$$-i\chi(\nu)' = \chi(\nu)(U - \nu H) \quad (13)$$

$$-i\dot{\chi}(\nu) = \chi(\nu)(UH + HU - \nu H^2) \quad (14)$$

each of them playing a role of a Lax pair for (6).

Consider for the moment the following general Zakharov-Shabat problems

$$i\partial\varphi(\mu) = (V - \mu J)\varphi(\mu) \quad (15)$$

$$-i\partial\psi(\lambda) = \psi(\lambda)(V - \lambda J) \quad (16)$$

$$-i\partial\chi(\nu) = \chi(\nu)(V - \nu J) \quad (17)$$

where  $\partial$  denotes a derivative with respect to some parameter. We will take the binary transformation in the form

$$\begin{aligned} \psi[1](\lambda, \mu, \nu) &= \psi(\lambda) \left[ \mathbf{1} - \frac{\nu - \mu}{\lambda - \mu} \varphi(\mu) (p\chi(\nu)\varphi(\mu)p)^{-1} \chi(\nu) \right] \\ &=: \psi(\lambda) \left[ \mathbf{1} - \frac{\nu - \mu}{\lambda - \mu} P \right] \end{aligned} \quad (18)$$

$$=: \psi(\lambda) \left[ \mathbf{1} - \frac{\nu - \mu}{\lambda - \mu} P \right] \quad (19)$$

where  $p$  is a constant projector ( $\partial p = 0$ ) and the inverse means an inverse in the  $p$ -invariant subspace:  $(pxp)^{-1}pxp = pxp(pxp)^{-1} = p$ . The operator  $P$  defined by (19) is idempotent ( $P^2 = P$ ) but in general non-Hermitian.  $P$  satisfies the nonlinear master equation [37]

$$i\partial P = (V - \mu J)P - P(V - \nu J) + (\mu - \nu)PJP \quad (20)$$

The binary transformation implies the following transformation of the potential

$$V[1](\mu, \nu) = V + (\mu - \nu)[P, J]. \quad (21)$$

Applying this general result to  $V = U$ ,  $J = H$  we get

$$U[1](\mu, \nu) = U + (\mu - \nu)[P, H]. \quad (22)$$

The second triple of equations we have started with corresponds to  $V = UH + HU$  and  $J = H^2$ . In this case

$$V[1](\mu, \nu) = U[1](\mu, \nu)H + HU[1](\mu, \nu). \quad (23)$$

This means that (22) guarantees simultaneous covariance of the Lax pairs under the binary transformation (18).

Another important feature of the binary transformation is the fact that for  $\nu = \bar{\mu}$  and  $p\chi(\bar{\mu}) = p\varphi(\mu)^\dagger$  the Hermiticity of the potential is Darboux-covariant, i.e

$$U[1](\mu, \bar{\mu})^\dagger = U[1](\mu, \bar{\mu}) \quad (24)$$

if  $U^\dagger = U$ .

Using (20) one can show by a straightforward calculation (see the Appendix) that the binary transformed

$$\psi[1] = \psi \left( \mathbf{1} - \frac{\bar{\mu} - \mu}{\lambda - \mu} P \right) \quad (25)$$

indeed satisfies

$$-i\psi[1]' = \psi[1](U[1] - \lambda H) \quad (26)$$

$$-i\dot{\psi}[1] = \psi[1](U[1]H + HU[1] - \lambda H^2) \quad (27)$$

with  $U[1] = U[1](\mu, \bar{\mu})$  and, therefore,

$$i\dot{U}[1] = [H, U[1]^2] + iU[1]'H + iHU[1]'. \quad (28)$$

Subsequent iterations of the Darboux transformation generate further solutions. Starting with a Hermitian solution  $U$  we obtain an infinite sequence of Hermitian solutions  $U[1], U[2], \dots$  satisfying  $\text{Tr } U = \text{Tr } U[1] = \text{Tr } U[2] \dots$

### III. COVARIANCE OF THE CONSTRAINT $U'H + HU' = 0$

In order to generate solutions of  $i\dot{U} = [H, U^2]$  one has to maintain the constraint  $U'H + HU' = U[1]'H + HU[1]' = U[2]'H + HU[2]' = \dots = 0$ . Starting with stationary solutions

$$i\varphi(\mu)' = z\varphi(\mu) \quad (29)$$

one finds that  $U' = 0$  implies  $U[1]' = 0$ . An alternative approach can be applied to Hamiltonians of the Dirac type

$$H = \mathbf{p} \cdot \boldsymbol{\alpha} + m\beta, \quad (30)$$

which satisfy  $H^2 = E(p)^2 \mathbf{1}$  and therefore imply

$$\begin{aligned} U[1]'H + HU[1]' &= U'H + HU' + (\mu - \bar{\mu})[P', H^2] \\ &= U'H + HU' \end{aligned} \quad (31)$$

which makes the constraint hereditary.

In general, using (20), one finds that the constraint is hereditary if

$$[P_{\perp}(U - \mu H)P - P(U - \nu H)P_{\perp}, H^2] = 0, \quad (32)$$

where  $P_{\perp} = \mathbf{1} - P$ .

### IV. PARTICULAR CASES

In this section we shall discuss properties of solutions corresponding to several choices of the initial  $U$ . For some applications one can restrict the general form (18) by choosing

$$p = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad \varphi(\mu) = \begin{pmatrix} \varphi_1 & 0 & \dots & 0 \\ \varphi_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_n & 0 & \dots & 0 \end{pmatrix}, \quad (33)$$

$\nu = \bar{\mu}$  and  $\chi(\bar{\mu}) = \varphi(\mu)^{\dagger}$ . Denoting the first column in  $\varphi(\mu)$  by  $|\varphi\rangle$  one finds that

$$U[1](\mu, \bar{\mu}) = U + (\mu - \bar{\mu})[P, H] \quad (34)$$

where

$$P = \frac{|\varphi\rangle\langle\varphi|}{\langle\varphi|\varphi\rangle}. \quad (35)$$

The transition from the column solution  $|\varphi\rangle$  to (33) is a useful trick that allows one to consider expressions such as  $p\varphi p$  which otherwise would not make any sense.

$$\mathbf{A.} \ U^2 = U, \ U' = 0$$

This is an interesting case since  $U = U(t)$  is a solution of the ordinary linear LvNE:

$$U(t) = \exp[-iHt]U(0)\exp[iHt]. \quad (36)$$

Let us take a solution stationary with respect to  $\tau$ :

$$i|\varphi(\mu)\rangle' = (U - \mu H)|\varphi(\mu)\rangle = z_\mu|\varphi(\mu)\rangle \quad (37)$$

and define

$$|\tilde{\varphi}\rangle = \exp[iHt]|\varphi\rangle. \quad (38)$$

The Lax pair is now

$$z_\mu|\tilde{\varphi}\rangle = (U(0) - \mu H)|\tilde{\varphi}\rangle \quad (39)$$

$$i|\dot{\tilde{\varphi}}\rangle = \frac{1}{\mu}(z_\mu - z_\mu^2)|\tilde{\varphi}\rangle \quad (40)$$

with the solution

$$|\varphi(t, \tau)\rangle = e^{-i(Ht + \alpha_{t, \tau})}|\varphi(0, 0)\rangle \quad (41)$$

where  $\alpha_{t, \tau} = \frac{1}{\mu}z_\mu(1 - z_\mu)t + z_\mu\tau$ . The projector (35) is  $\tau$ -independent and satisfies the *linear* LvNE. This implies that  $U[1]$  satisfies the same linear equation as  $U$ . The following lemmas explain the origin of this effect. Consider the general  $P$  defined by (19) and  $V[1] = V[1](\mu, \nu)$ .

**Lemma 1.**  $\partial P = 0$  implies

$$\begin{aligned} V[1]^2 = V^2 + (\mu - \nu) & \left( P(JV + VJ - \nu J^2)P_\perp \right. \\ & \left. - P_\perp(JV + VJ - \mu J^2)P \right). \end{aligned} \quad (42)$$

*Proof:* (20) implies

$$P(V - \nu J) - (V - \mu J)P = (\mu - \nu)PJP \quad (43)$$

and

$$\begin{aligned} (\mu - \nu)(PJPJ + JPJP - JPJ) \\ = [P, V]J + J[P, V] + \mu J^2P - \nu PJ^2. \end{aligned} \quad (44)$$

The latter formula leads directly to (42).  $\square$

**Lemma 2.** Assume  $P' = 0$  and  $\dot{P}$  is given by (20) with  $V = HU + UH$ ,  $J = H^2$ . Then

$$U[1]^2 = U^2 - (\mu - \nu)i\dot{P}. \quad (45)$$

*Proof:*

$$-iP_\perp\dot{P} = -P_\perp(UH + HU - \mu H^2)P \quad (46)$$

$$-i\dot{P}P_\perp = P(UH + HU - \nu H^2)P_\perp \quad (47)$$

and therefore

$$\begin{aligned} U[1]^2 &= U^2 - i(\mu - \nu)(P_\perp\dot{P} + \dot{P}P_\perp) \\ &= U^2 - i(\mu - \nu)\dot{P}. \end{aligned}$$

$\square$

An immediate consequence of Lemma 2 is

**Lemma 3.** Assume  $P' = 0$  and  $U^2 = U$ . Then  $U[1]^2 = U[1]$  if and only if  $i\dot{P} = [H, P]$  i.e.  $P$  satisfies the linear LvNE.

**B. Case**  $U^2 - aU \neq \text{const} \cdot \mathbf{1}$ ,  $[U^2 - aU, H] = 0$ ,  $U' = 0$

Let  $a$  be a real number.  $[U^2 - aU, H] = 0$  implies

$$U(t) = \exp[-iaHt]U(0)\exp[iaHt]. \quad (48)$$

Repeating the steps from the previous subsection we obtain the Lax pair

$$z_\mu|\tilde{\varphi}\rangle = (U(0) - \mu H)|\tilde{\varphi}\rangle \quad (49)$$

$$i|\dot{\tilde{\varphi}}\rangle = \frac{1}{\mu}(\Delta_a + az_\mu - z_\mu^2)|\tilde{\varphi}\rangle \quad (50)$$

where  $\Delta_a = U(0)^2 - aU(0)$ . The projector  $P$  is  $\tau$ -independent but possesses a nontrivial  $t$ -dependence which follows from the fact that  $\mu - \bar{\mu} \neq 0$ . Define the function

$$F_a(t) = \langle\varphi(0,0)|\exp\left(i\frac{\mu - \bar{\mu}}{|\mu|^2}\Delta_a t\right)|\varphi(0,0)\rangle \quad (51)$$

satisfying

$$\langle\varphi(t,\tau)|\varphi(t,\tau)\rangle = \exp[i(\bar{\alpha}_{t,\tau} - \alpha_{t,\tau})]F_a(t), \quad (52)$$

where  $\alpha_{t,\tau} = \frac{1}{\mu}z_\mu(a - z_\mu)t + z_\mu\tau$ . We find finally

$$U[1](t) = e^{-iaHt}\left(U(0) + (\mu - \bar{\mu})F_a(t)^{-1} \quad (53)$$

$$\begin{aligned} &\times e^{-\frac{i}{\mu}\Delta_a t} [|\varphi(0,0)\rangle\langle\varphi(0,0)|, H] e^{\frac{i}{\bar{\mu}}\Delta_a t} e^{iaHt} \\ &=: e^{-iaHt}U_{\text{int}}(t)e^{iaHt}. \end{aligned} \quad (54)$$

Let us note that what makes  $U[1](t)$  nontrivial is essentially the presence of  $F(t)$  in the denominator. It is precisely this property of the binary Darboux transformation that is responsible for the soliton solutions in the Maxwell-Bloch case [9].

## V. EXAMPLES

We shall now demonstrate on explicit examples how the method works. We will concentrate on the first Darboux transformation  $U[1]$ . Further iterations,  $U[2]$ , ...,  $U[n]$ , are also interesting and their relation to  $U[1]$  is similar to this between solitons and multi-solitons. The problem will be discussed in a forthcoming paper.

### A. $3 \times 3$ matrix Hamiltonian, $a = 1$

Consider the Hamiltonian

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (55)$$

and take  $\mu = i$  (for real  $\mu$  the binary transformation is trivial). We begin with

$$U(0) = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{1}{2} - \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \quad (56)$$

$U(0)$  does not commute with  $H$  but

$$U(0)^2 - U(0) = U(t)^2 - U(t) = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (57)$$

does. The eigenvalues of  $U(0) - iH$  are  $z_{\pm} = (1 \pm i\sqrt{2})/2$  and  $z_-$  has degeneracy 2. The two orthonormal eigenvectors corresponding to  $z_-$  are

$$|\varphi_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\varphi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} \\ 1 \\ 0 \end{pmatrix}. \quad (58)$$

Taking

$$|\varphi(0, 0)\rangle = \frac{1}{\sqrt{2}}(|\varphi_1\rangle + |\varphi_2\rangle) \quad (59)$$

we get  $F(t) = \cosh(t/2)$  and the internal part defined by (54) is given explicitly by

$$U_{\text{int}}(t) = \begin{pmatrix} \frac{1+\sqrt{2}}{2} - \frac{\sqrt{2}}{1+e^t} & 0 & \frac{-1-i}{2\sqrt{2}\cosh(t/2)} \\ 0 & \frac{1-\sqrt{2}}{2} + \frac{\sqrt{2}}{1+e^t} & \frac{1}{2\cosh(t/2)} \\ \frac{-1+i}{2\sqrt{2}\cosh(t/2)} & \frac{1}{2\cosh(t/2)} & \frac{1}{2} \end{pmatrix}. \quad (60)$$

One can check by an explicit calculation that (54) with (55) and (60) is a Hermitian solution of

$$i\dot{U}[1] = [H, U[1]^2]. \quad (61)$$

Let us note that the solution (60) corresponds to an initial condition  $U[1](0)$  which is different from  $U(0)$  and is no longer block-diagonal in the basis block-diagonalizing  $H$ . This is a consequence of the fact that  $P$  is not block diagonal, a fact that explains the importance of the degeneracy condition for  $z_-$  (had we chosen  $z_+$  we would have obtained a  $(2 \times 2) \oplus 1$  block-diagonal  $P$ ). The eigenvalues of  $U[1](t)$  are nevertheless the same as those of  $U(0)$ . This follows immediately from the  $t$ -independence of spectrum of  $U[1](t)$  and the fact that  $U[1](t)$  tends asymptotically to  $U(t)$  for  $t \rightarrow +\infty$ . As a consequence  $U[1](t)$  is neither normalized ( $\text{Tr}U[1] \neq 1$ ) nor positive and hence cannot be regarded as a density matrix. It is, however, very easy to obtain a density matrix solution once we know  $U[1](t)$ . The problem reduces to generating a new solution whose spectrum is shifted with respect to the original one by a number. This can be accomplished by a gauge transformation. Indeed,

$$\tilde{U}[1] = e^{-2i\lambda H t} (U[1] + \lambda \mathbf{1}) e^{2i\lambda H t} \quad (62)$$

is also a solution of (61), and its spectrum is shifted by  $\lambda$  with respect to this of  $U[1]$ . Such positive solutions can be regarded as non-normalized density matrices and are sufficient for a well defined probability interpretation of the theory. Let us finally note that the fact that spectrum of a Hermitian solution is conserved by the dynamics is not accidental but follows from general properties of Lie-Nambu equations [17].

### B. $3 \times 3$ Hamiltonian with equally-spaced spectrum

Consider the Hamiltonian ( $k, m \in \mathbf{R}$ )

$$H = \begin{pmatrix} k+m & -m & 0 \\ -m & k+m & 0 \\ 0 & 0 & k+m \end{pmatrix}, \quad (63)$$

whose eigenvalues are  $k, k+m, k+2m$ , and take  $\mu = i$ . We begin with a non-normalized density matrix

$$\rho(0) = \begin{pmatrix} \frac{1}{2}(a + \sqrt{4b + a^2}) & 0 & 0 \\ 0 & \frac{1}{2}(a - \sqrt{4b + a^2}) & 0 \\ 0 & 0 & c \end{pmatrix}, \quad (64)$$

satisfying

$$\rho(0)^2 - a\rho(0) = \rho(t)^2 - a\rho(t) = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c(c-a) \end{pmatrix}. \quad (65)$$

Eigenvalues of  $\rho(0) - iH$  are  $z_0 = c - i(k+m)$ ,  $z_{\pm} = \frac{1}{2}(a \pm \sqrt{a^2 + 4(b-m^2)}) - i(k+m)$ . We need this spectrum to satisfy a degeneracy condition:  $z_0 = z_+$  or  $z_0 = z_-$  with  $c$  real and non-negative. Positivity of  $\rho(0)$  requires also that  $a > 0$ ,  $a - \sqrt{4b+a^2} \geq 0$ . We will require that  $b \neq 0$  (otherwise we will not get a nontrivial  $\rho[1]$ ) so that the parameters finally satisfy  $0 < 4m^2 < a^2 + 4b < a^2$ . Let us note that  $c(c-a) = b - m^2$  independently of the choice of sign in the degeneracy condition  $z_0 = z_{\pm}$ .

Denote by  $|k+m\rangle$  the joint eigenstate of  $H$  (with eigenvalue  $k+m$ ) and  $\rho(0) - iH$  (with eigenvalue  $c - i(k+m)$ ); the corresponding projector is  $P_{k+m} = |k+m\rangle\langle k+m|$ . Let  $\mathbf{1}_3 = |k\rangle\langle k| + |k+m\rangle\langle k+m| + |k+2m\rangle\langle k+2m|$ , where the three projectors project on eigenstates of  $H$ . We can write

$$\Delta_a = b\mathbf{1}_3 - m^2 P_{k+m}. \quad (66)$$

The two eigenstates corresponding to the degenerate eigenvalue  $c - i(k+m)$  are orthogonal. One of them is simply  $|\varphi_1\rangle = |k+m\rangle$ ; the other one is  $|\varphi_2\rangle = \phi_k|k\rangle + \phi_{k+2m}|k+2m\rangle$ , where the explicit form of  $\phi_j$  is for the moment irrelevant. Now take  $|\varphi(0,0)\rangle = A|\varphi_1\rangle + B|\varphi_2\rangle$ ,  $|A|^2 + |B|^2 = 1$ . We find

$$F_a(t) = e^{-2bt} \left( 1 + (e^{2m^2t} - 1)|A|^2 \right). \quad (67)$$

The above formulas can be also written as

$$H = \sum_{n=0,m,2m} (k+n)|k+n\rangle\langle k+n| \quad (68)$$

$$\begin{aligned} \rho(0) &= \frac{a}{2} \left( |k\rangle\langle k| + |k+2m\rangle\langle k+2m| \right) + c|k+m\rangle\langle k+m| \\ &\quad - \frac{1}{2}\sqrt{4b+a^2} \left( |k+2m\rangle\langle k| + |k\rangle\langle k+2m| \right). \end{aligned} \quad (69)$$

The solution is

$$\begin{aligned} \rho[1](t) &= \rho(t) + 2im \left( 1 + (e^{2m^2t} - 1)|A|^2 \right)^{-1} \\ &\quad \times \left( |B|^2 (\bar{\phi}_{k+2m} + \bar{\phi}_k)(\phi_{k+2m} - \phi_k) |k, t\rangle\langle k+2m, t| \right. \\ &\quad + \frac{1}{\sqrt{2}} e^{m^2t} A \bar{B} (\bar{\phi}_{k+2m} + \bar{\phi}_k) |k, t\rangle\langle k+m, t| \\ &\quad + \frac{1}{\sqrt{2}} e^{m^2t} A \bar{B} (\bar{\phi}_{k+2m} - \bar{\phi}_k) |k+2m, t\rangle\langle k+m, t| \\ &\quad \left. - \text{H.c.} \right) \end{aligned} \quad (70)$$

where  $|k+j, t\rangle = e^{-iajt}|k+j\rangle$ .

### C. 1-dimentional harmonic oscillator

We begin with the Hamiltonian

$$H = \sum_{n=0}^{\infty} \hbar\omega \left( \frac{1}{2} + n \right) \left| \frac{1}{2} + n \right\rangle \left\langle \frac{1}{2} + n \right|. \quad (71)$$

One can directly apply the construction from the above example. We have to choose some three-dimensional subspace which defines  $\rho(0)$ . Put  $k = \frac{1}{2} + l$  ( $l, m \in \mathbf{N}$ ), and  $\mu = i/(\hbar\omega)$ . The solution is

$$\begin{aligned}
\rho[1](t) = & \rho(t) + 2im \left( 1 + (e^{2\omega m^2 t} - 1)|A|^2 \right)^{-1} \\
& \times \left( |B|^2 (\bar{\phi}_{k+2m} + \bar{\phi}_k)(\phi_{k+2m} - \phi_k) |k, t\rangle \langle k+2m, t| \right. \\
& + \frac{1}{\sqrt{2}} e^{\omega m^2 t} A \bar{B} (\bar{\phi}_{k+2m} + \bar{\phi}_k) |k, t\rangle \langle k+m, t| \\
& + \frac{1}{\sqrt{2}} e^{\omega m^2 t} A \bar{B} (\bar{\phi}_{k+2m} - \bar{\phi}_k) |k+2m, t\rangle \langle k+m, t| \\
& \left. - \text{H.c.} \right)
\end{aligned} \tag{72}$$

$\rho[1]$  has interesting asymptotic properties. Assume  $A \neq 0$ . For  $t \gg 0$   $\rho[1](t) \approx \rho(t)$  which suggests that the nonlinear effect is transient. However, for  $t \ll 0$

$$\rho[1](t) \approx \rho(t) + 2im \left( (\bar{\phi}_{k+2m} + \bar{\phi}_k)(\phi_{k+2m} - \phi_k) |k, t\rangle \langle k+2m, t| - \text{H.c.} \right). \tag{73}$$

It follows that the asymptotic dynamics of  $\rho[1](t)$  is linear but around  $t = 0$  some sort of “phase transition” occurs, and the result of this transition is stable. Let us also note that the linear evolution is determined by  $\exp(-iaHt)$  with  $|a| > 2m$  and  $m \in \mathbf{N}$ . The choice of  $a$  is related to the initial condition. We obtain, therefore, an effective nonlinear modification of frequency of the oscillator.

Let us finally make  $\phi_j$  explicit. Assume  $l = 0$ ,  $m = 1$ ,  $a = 5$ ,  $b = -4$ ,  $z_0 = z_+$  (i.e.  $c = (5 + \sqrt{5})/2$ ),  $A = B = 1/\sqrt{2}$ . Now

$$\rho(0) = \frac{5}{2} \left( \left| \frac{1}{2} \right\rangle \left\langle \frac{1}{2} \right| + \left| \frac{5}{2} \right\rangle \left\langle \frac{5}{2} \right| \right) + \frac{5 + \sqrt{5}}{2} \left| \frac{3}{2} \right\rangle \left\langle \frac{3}{2} \right| - \frac{3}{2} \left( \left| \frac{5}{2} \right\rangle \left\langle \frac{1}{2} \right| + \left| \frac{1}{2} \right\rangle \left\langle \frac{5}{2} \right| \right) \tag{74}$$

$$|\varphi_1\rangle = \left| \frac{3}{2} \right\rangle \tag{75}$$

$$|\varphi_2\rangle = -i \sqrt{\frac{3 + \sqrt{5}}{6}} \left| \frac{1}{2} \right\rangle + \sqrt{\frac{2}{9 + 3\sqrt{5}}} \left| \frac{5}{2} \right\rangle. \tag{76}$$

$\text{Tr } \rho(0) = (15 + \sqrt{5})/2$  and the eigenvalues of  $\rho(0)$  are 4, 1, and  $(5 + \sqrt{5})/2$ .

#### D. Linear equation with nonlinear perturbation

Assume  $i\dot{\rho}_\epsilon = \epsilon[H, \rho_\epsilon^2]$  and define

$$\rho = \exp[-iHt] \rho_\epsilon \exp[iHt]. \tag{77}$$

Then

$$i\dot{\rho} = [H, \rho] + \epsilon[H, \rho^2]. \tag{78}$$

This is a Nambu-type equation obtained by taking a linear Hamiltonian function  $H_f = \text{Tr}(H\rho)$  and  $S = \text{Tr}(\rho^2)/2 + \epsilon \text{Tr}(\rho^3)/3$ . Average energy is, by definition  $\langle H \rangle = \text{Tr}(H\rho)/\text{Tr } \rho$ .

Returning to the example of the harmonic oscillator we proceed as before but now we choose  $\mu = i/(\epsilon\hbar\omega)$ . The solution becomes

$$\begin{aligned}
\rho[1](t) = & e^{-i(1+a\epsilon)Ht} \left[ \rho(0) + 2im \left( 1 + (e^{2\epsilon\omega m^2 t} - 1)|A|^2 \right)^{-1} \right. \\
& \times \left( |B|^2 (\bar{\phi}_{k+2m} + \bar{\phi}_k)(\phi_{k+2m} - \phi_k) |k\rangle \langle k+2m| \right. \\
& + \frac{1}{\sqrt{2}} e^{\epsilon\omega m^2 t} A \bar{B} (\bar{\phi}_{k+2m} + \bar{\phi}_k) |k\rangle \langle k+m| \\
& + \frac{1}{\sqrt{2}} e^{\epsilon\omega m^2 t} A \bar{B} (\bar{\phi}_{k+2m} - \bar{\phi}_k) |k+2m\rangle \langle k+m| \\
& \left. \left. - \text{H.c.} \right) \right] e^{i(1+a\epsilon)Ht}.
\end{aligned} \tag{79}$$

The asymptotic dynamics is again linear and the frequency shift is  $\Delta\omega = a\epsilon\omega$ . Let us note that according to the definition of  $\langle H \rangle$  the eigenvalues of energy should be assumed to take values  $\hbar\omega(1/2+n)$  and not  $(1+a\epsilon)\hbar\omega(1/2+n)$ .

This point is essential for the probability interpretation of such a nonlinear theory.

### E. Homogeneous modification of the equation

The equation we have solved is non-homogeneous which implies that  $\rho \mapsto \text{const} \cdot \rho$  is not a symmetry transformation. This fact makes it necessary to work with non-normalized density matrices. In order to obtain a homogeneous equation one can utilize the fact that  $\text{Tr}(\rho^n)$  is time-independent (as a Casimir invariant). Define  $C(\rho) = [\text{Tr} \rho / \text{Tr}(\rho^3)]^{1/2}$  and consider

$$i\dot{\rho} = C(\rho)[H, \rho^2]. \quad (80)$$

The equation is 1-homogeneous in  $\rho$  and its solutions can be obtained by the substitution  $t \mapsto C(\rho)t$  in the corresponding formulas given above. The multiplication of  $\rho$  by constants is a symmetry operation so that we can easily produce solutions satisfying  $\text{Tr} \rho = 1$ . To get the equation from the Nambu-type formalism one takes  $S(\rho) = \frac{2}{3}[\text{Tr} \rho \text{Tr}(\rho^3)]^{1/2}$ .

### F. Two spin-1/2 particles

The above Nambu-type formalism implies that spectra of Hermitian solutions are time-independent. In particular, assuming that the nonlinear dynamics is defined for a two-particle system, the corresponding two-particle density matrix has time-independent eigenvalues. When it comes to *reduced* density matrices of the one-particle subsystems the situation is less simple. Assume the two-particle system is described by the Hamiltonian

$$H = H_1 \otimes \mathbf{1} + \mathbf{1} \otimes H_2. \quad (81)$$

On the one hand it is clear that traces of the reduced density matrices are time-independent. On the other hand, it can be shown [18] that

$$i \frac{d}{dt} \text{Tr}_1 \left( (\text{Tr}_2 \rho)^2 \right) = 2 \text{Tr}_1 \left( [\text{Tr}_2(\rho^2), \text{Tr}_2(\rho)] H_1 \right), \quad (82)$$

where  $\text{Tr}_k$ ,  $k = 1, 2$  are partial traces. For  $\rho^2 \neq \rho$  the RHS of (82) does not in general vanish and this means that the eigenvalues of the reduced density matrix  $\text{Tr}_2 \rho$  can be time-dependent. What is interesting the average energies of the subsystems do not change as both  $\text{Tr} H_1 \otimes \mathbf{1} \rho$  and  $\text{Tr} \mathbf{1} \otimes H_2 \rho$  are separately conserved. It follows that although the two subsystems do not exchange average energy, they nevertheless exhibit some kind of collective behavior. Since it is difficult to investigate the effect from a general perspective, it may be instructive to consider an explicit example of a two-particle system whose density matrix can be explicitly calculated by the Darboux technique.

Consider two spin-1/2 particles described by the Hamiltonian

$$H = \boldsymbol{\sigma} \cdot \mathbf{a} \otimes \mathbf{1} + \mathbf{1} \otimes \boldsymbol{\sigma} \cdot \mathbf{b}. \quad (83)$$

To make the example concrete assume that  $|\mathbf{b}| = 1$  and  $|\mathbf{a}| = 2$ . We will start with the non-normalized density matrix

$$\rho(0) = \frac{1}{2} \begin{pmatrix} 5 + \sqrt{7} & 0 & 0 & 0 \\ 0 & 5 - \sqrt{7} & 0 & 0 \\ 0 & 0 & 5 + \sqrt{15} & 0 \\ 0 & 0 & 0 & 5 - \sqrt{15} \end{pmatrix} \quad (84)$$

which is written in such a basis that

$$H = 2\sigma_x \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_z = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1 \end{pmatrix}. \quad (85)$$

Take  $a = 5$ . We find

$$\Delta_5 = \rho(0)^2 - 5\rho(0) = -\frac{1}{2} \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \quad (86)$$

so that  $[\Delta_5, H] = 0$ . Taking  $\mu = i$  we find that  $\rho(0) - iH$  has eigenvalues  $z_1 = (1+i)/2$ ,  $z_2 = (1+3i)/2$ ,  $z_3 = (1-5i)/2$ , where  $z_1$  has degeneracy 2. The two eigenvectors corresponding to  $z_1$  are

$$|\varphi_1\rangle = \frac{1}{4\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 + i\sqrt{15} \\ 4 \end{pmatrix}, \quad |\varphi_2\rangle = \frac{1}{4\sqrt{2}} \begin{pmatrix} -3 + i\sqrt{7} \\ 4 \\ 0 \\ 0 \end{pmatrix}. \quad (87)$$

Assuming

$$|\varphi(0, 0)\rangle = \frac{1}{\sqrt{2}} (|\varphi_1\rangle + |\varphi_2\rangle). \quad (88)$$

we obtain

$$F_5(t) = \frac{1}{2} (e^{5t} + e^{9t}) \quad (89)$$

and  $\rho[1](t) = \exp[-5iHt]\rho_{\text{int}}(t)\exp[5iHt]$  where

$$\rho_{\text{int}}(t) = \frac{1}{2} \begin{pmatrix} 5 - \sqrt{7} \tanh 2t & 0 & \frac{-13i - 3\sqrt{7} - \sqrt{15} - i\sqrt{105}}{8 \cosh 2t} & \frac{-7i + 3\sqrt{7} - 3\sqrt{15} + i\sqrt{105}}{8 \cosh 2t} \\ 0 & 5 + \sqrt{7} \tanh 2t & \frac{15i + \sqrt{7} - \sqrt{15} - i\sqrt{105}}{8 \cosh 2t} & \frac{\sqrt{7} + \sqrt{15}}{2 \cosh 2t} \\ \frac{13i - 3\sqrt{7} - \sqrt{15} + i\sqrt{105}}{8 \cosh 2t} & \frac{-15i + \sqrt{7} - \sqrt{15} + i\sqrt{105}}{8 \cosh 2t} & 5 + \sqrt{15} \tanh 2t & 0 \\ \frac{7i + 3\sqrt{7} - 3\sqrt{15} - i\sqrt{105}}{8 \cosh 2t} & \frac{\sqrt{7} + \sqrt{15}}{2 \cosh 2t} & 0 & 5 - \sqrt{15} \tanh 2t \end{pmatrix}. \quad (90)$$

Eigenvalues  $p_{\pm}(k)$ ,  $k = 1, 2$ , of (normalized) reduced density matrices of the  $k$ -th subsystems are

$$p_{\pm}(1) = \frac{1}{2} \pm \frac{\sqrt{15} - \sqrt{7}}{20} \tanh 2t \quad (91)$$

$$p_{\pm}(2) = \frac{1}{2} \pm \frac{\sqrt{26 + 2\sqrt{105}}}{40 \cosh 2t}. \quad (92)$$

In order to check that (82) is indeed satisfied one has to use non-normalized density matrices (since the equation is non-homogeneous) i.e. put  $\text{Tr}_1((\text{Tr}_2 \rho)^2) = 100(p_+(1)^2 + p_-(1)^2)$ . Average energies of both subsystems are 0 for any  $t$ , which also agrees with general theorems.

The above collective phenomenon is typical of higher-entropy dynamics and does not occur in Hartree-type equations [19], a fact that explicitly shows that the Nambu-type dynamics exhibits properties essentially different from those discussed in the context of completely positive nonlinear maps in [20]. This interesting problem requires further studies and is beyond the scope of nonrelativistic theory.

## VI. CONCLUSIONS

We have proposed an algebraic technique of solving a nonlinear operator equation. The equation we have discussed can be regarded as a Heisenberg-picture equation of motion for an operator  $U$ , since writing it in the form

$$i\dot{U} = [H, U^2] = [HU + UH, U] \quad (93)$$

one obtains a nonlinear Heisenberg equation with the time-dependent Hamiltonian  $\tilde{H}(U) = -HU - UH$ . The choice of non-Hermitian  $U$  (typical of the binary transformation with  $\nu \neq \bar{\mu}$ ) leads to non-Hermitian  $\tilde{H}$ , a fact that may be of interest for a theory of open systems.

Restricting the initial solution  $U$  to projectors ( $U^2 = U$ ) we have shown that there exists a linear orbit of the Darboux-transformation ( $U[1]^2 = U[1]$  and, hence,  $U[1]$  is a solution of the *linear* LvNE). This shows incidentally

that the binary transformation can be used to generate solutions of the ordinary linear LvNE, a property that may find applications in other contexts.

Looking more closely at the origin of the simultaneous covariance of both equations constituting the Lax pair, one can immediately write other Lax pairs whose compatibility conditions provide new nonlinear Darboux-integrable operator equations. For example, taking the second equation with  $V = H^2U + HUH + UH^2$ ,  $J = H^3$ , and assuming the constraint  $U' = 0$  one obtains the compatibility condition

$$i\dot{U} = [H^2U + HUH + UH^2, U]. \quad (94)$$

This highly non-Abelian nonlinear Liouville-von Neumann (or Heisenberg) equation can be solved by the binary Darboux transformation in a way similar to this described above.

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## VII. APPENDIX: PROOF OF DARBOUX COVARIANCE

We will show that (19) satisfies

$$-i\partial\psi[1] = \psi[1](V[1] - \lambda J) \quad (95)$$

with  $V[1]$  given by (21):

$$\begin{aligned} & -i\partial\psi[1](\lambda, \mu, \nu) \\ &= \psi(\lambda)(V - \lambda J) \left[ \mathbf{1} - \frac{\nu - \mu}{\lambda - \mu} P \right] + \frac{\nu - \mu}{\lambda - \mu} \psi(\lambda) \\ & \quad \times \left[ (V - \mu J)P - P(V - \nu J) + (\mu - \nu)PJP \right] \\ &= \psi[1](\lambda, \mu, \nu)(V - \lambda J) + \frac{\nu - \mu}{\lambda - \mu} \psi(\lambda) \\ & \quad \times \left[ -(V - \lambda J)P + (V - \mu J)P - (\lambda - \nu)PJ \right. \\ & \quad \left. + (\mu - \nu)PJP \right] \\ &= \psi[1](\lambda, \mu, \nu)(V - \lambda J) + \frac{\nu - \mu}{\lambda - \mu} \psi(\lambda) \\ & \quad \times \left[ (\lambda - \mu)JP - (\lambda - \nu)PJ + (\mu - \nu)PJP \right] \\ &= \psi[1](\lambda, \mu, \nu)(V[1] - \lambda J). \end{aligned}$$

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$$\partial(pxp)^{-1} = -(pxp)^{-1}\partial(pxp)(pxp)^{-1}.$$